Chapter 3 - Day 3

2 fundamental notions in Calculus:

Continuity and Differentiability

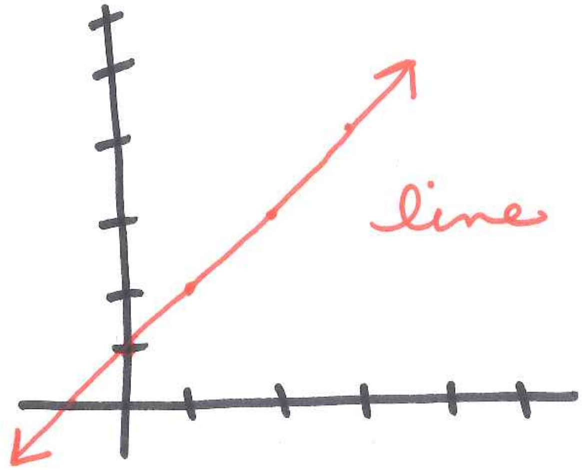
a function f is <u>continuous at a point</u> $x=c$ if

$$\lim_{x \to c} f(x) = f(c)$$

a function f is <u>continuous on an interval</u> if it is continuous at every point of that interval.

Consider $f(x) = x+1$

$$\lim_{x \to 2} x+1 = \underset{f(2)}{\underset{\shortparallel}{2+1}} = 3$$



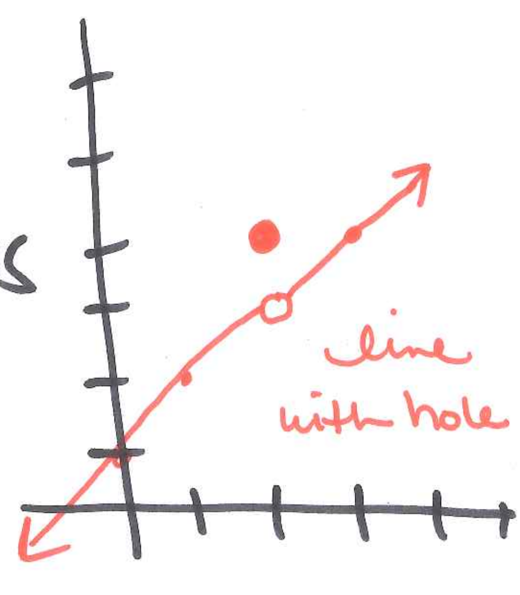
$f(x)$ is continuous at $x=2$

(in fact, continuous at all x.)

Consider $g(x) = \begin{cases} x+1 & x \neq 2 \\ 4 & x=2 \end{cases}$

$$\lim_{x \to 2} g(x) = 3 \neq g(2) = 4$$

So $g(x)$ is not continuous at $x=2$.



Graphically, a graph is continuous if there are no holes, jumps, or gaps at any point in the domain.
(We can draw the graph without picking up our pencil.)

<u>Fact</u>: if $f(x)$ and $g(x)$ are continuous functions at a point c, then the following are continuous at $x=c$ also.

$k f(x)$ where k is a constant.

$f(x)+g(x)$

$f(x) \cdot g(x)$

$\frac{f(x)}{g(x)}$ where $g(c) \neq 0$

<u>Note</u>: Polynomials are continuous at every point. Rational functions are continuous at every point in their domain.

<u>Ex</u>: Consider $f(x) = \begin{cases} x^2+2 & x \le 2 \\ 3x+A & x > 2 \end{cases}$

find A such that f(x) is continuous at x=2.

$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^+} f(x)$$

$$\lim_{x \to 2^-} x^2+2 = \lim_{x \to 2^+} 3x+A$$

* polynomials are continuous - use substitution!

$$(2)^2+2 = 3(2)+A$$

$$6 = 6+A$$

$$\boxed{0=A}$$

<u>Ex:</u> find B such that

$$f(x) = \begin{cases} x^3+1 & x<0 \\ 4x+B & x\geq 0 \end{cases}$$ is continuous

$$\lim_{x\to 0^-} x^3+1 = \lim_{x\to 0^+} 4x+B$$

$$0^3+1 = 4(0)+B$$

$$\boxed{1 = B}$$

a function f is <u>differentiable at a point</u> $x=c$ if

$$\lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = f'(c)$$

* hmm.... looks like "IRoC = derivative"

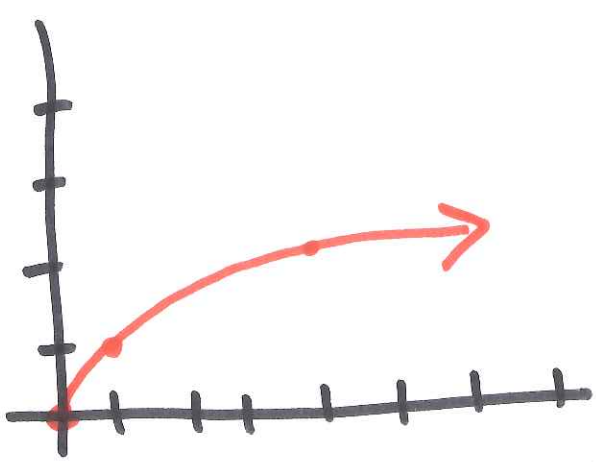
Graphically, if f is differentiable we can approximate f with a well-defined (non-vertical) tangent line

(This means the graph is smooth and does not have "sharp points/turns."

Note: Polynomials are differentiable at every point. Rational functions are differentiable at every point in their domain.

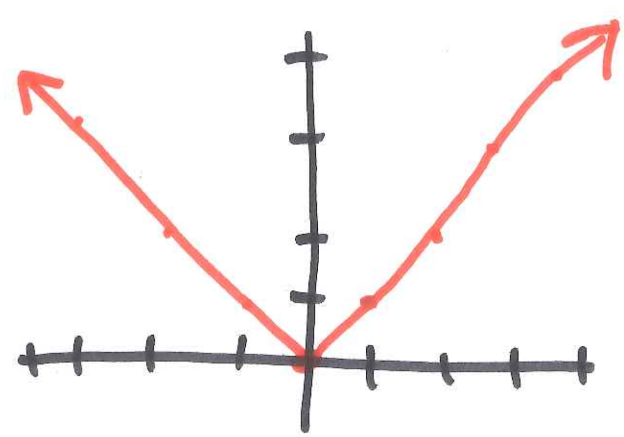
Ex: Consider the functions.

$f(x) = \sqrt{x}$



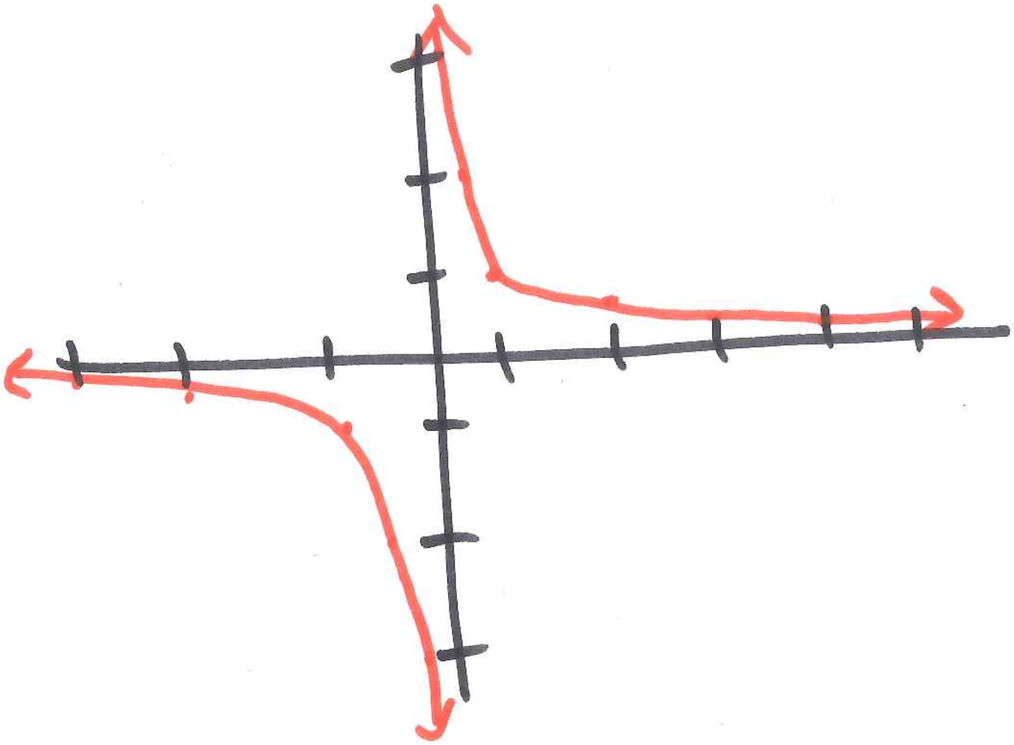
at $x=0$, vertical tangent line (not differentiable at $x=0$)

$f(x) = |x|$



Sharp turn at $x=0$ so not differentiable at $x=0$ but continuous at $x=0$.

Ex: $f(x) = \frac{1}{x}$



$f(x)$ not continuous at $x=0$

vertical asymptote = vertical tangent line so $f(x)$ not differentiable at $x=0$.

Theorem: • if f is differentiable at $x=c$, then f is also continuous at $x=c$.

• if f is not continuous at $x=c$, $f(x)$ is not differentiable at $x=c$.